

Fedosov supermanifolds: II. Normal coordinates

BODO GEYER ^{a) 1} and PETER LAVROV ^{a) b) 2}

^{a)} *Center of Theoretical Studies and Institute of Theoretical Physics,
Leipzig University, Augustusplatz 10/11, D-04109 Leipzig, Germany*

^{b)} *Tomsk State Pedagogical University, 634041 Tomsk, Russia*

The study of recently introduced Fedosov supermanifolds [1] is continued. Using normal coordinates, properties of even and odd symplectic supermanifolds endowed with a symmetric connection respecting given symplectic structure are studied.

1 Introduction

The formulation of fundamental physical theories, classical as well as quantum ones, by differential geometric methods nowadays is well established and has a great conceptual virtue. Probably, the most prominent example is the formulation of general relativity on Riemannian manifolds, i.e., the geometrization of the gravitational force; no less important is the geometric formulation of gauge field theories of the fundamental forces on fiber bundles. Another essential route has been opened by the formulation of classical mechanics – and also classical field theories – on symplectic manifolds and their connection with the (geometric) quantization. The properties of such kind of manifolds are widely studied.

Recently, some specific considerations in quantum field theory involve more complicated manifolds namely the so-called Fedosov manifolds, i.e., symplectic manifolds equipped with a symmetric connection which respects the symplectic structure. Fedosov manifolds have been introduced for the first time in the framework of deformation quantization [2]. The properties of Fedosov manifolds have been investigated in details (see, e.g., Ref. [3]). Especially, let us mention that for any Fedosov manifold the scalar curvature K is trivial, $K = 0$, and that the intrigue relation $\omega_{ij,kl} = (1/3)R_{klij}$, in terms of normal coordinates, holds between the symplectic structure ω_{ij} and the curvature tensor R_{klij} .

The discovery of supersymmetry [4] has introduced into modern quantum field theory the notion of supermanifolds proposed by Berezin [5]. Systematic considerations of supermanifolds and Riemannian supermanifolds were performed by DeWitt [6]. At present, even and odd symplectic supermanifolds and the corresponding differential geometry are widely involved and studied in consideration of some problems of modern theoretical and mathematical physics [7].

However, the situation concerning Fedosov supermanifolds is quite different. Only some specific problems concerning even Fedosov supermanifolds were under consideration. Especially, flat even Fedosov supermanifolds have been used in the study of a coordinate-free scheme of deformation quantization [10], for an explicit realization of the extended antibrackets [11] and for the formulation of modified triplectic quantization in general coordinates [12]. Deformation quantization for any even symplectic supermanifolds has been constructed in Ref. [13] (see also [14]). It was the main aim in [1] to start a systematic investigation of basic properties of

¹E-mail: geyer@itp.uni-leipzig.de

²E-mail: lavrov@tspu.edu.ru

even and odd Fedosov supermanifolds. In particular, some basic difference in even and odd Fedosov supermanifolds has been found which can be expressed in terms of the scalar curvature K . Namely, for any even Fedosov supermanifold the scalar curvature, as in usual differential geometry, is equal to zero while for odd Fedosov supermanifolds it is, in general, non-trivial. This fundamental fact has forced us to continue the investigation of the geometry of Fedosov supermanifolds. Thereby, without reviewing it here, definitions and results of [1] will be used where we need them.

The paper is organized as follows. In Sec. 2, we consider the properties of the Christoffel symbols in normal coordinates and derive an infinite system of equations for the affine extensions of the Christoffel symbols. In Sec. 3, the relation between the first order affine extension of Christoffel symbols and the symplectic curvature tensor both in normal and any local coordinates are studied. In Sec. 4, we derive a relation between the second order affine extension of the symplectic structure and the symplectic curvature tensor. Sec. 5 is devoted to the derivation of identities containing the first order affine extension of the symplectic curvature tensor and to a relation between the third order affine extension of symplectic structure and the first order affine extension of the symplectic curvature tensor. In Sec. 6, a short summary is given.

We use the condensed notations and the definition of tensor fields on supermanifolds as given by DeWitt [6]. Derivatives with respect to the coordinates x^i are understood as acting from the right and for them the notation $A_{,i} = \partial_r A / \partial x^i$ is used. Covariant derivatives also act from the right $A_{;i} = A \nabla_i$. The Grassmann parity of any quantity A is denoted by $\epsilon(A)$.

2 Affine extensions of Christoffel symbols and tensors on symplectic supermanifolds

Let us consider a $2N$ -dimensional symplectic supermanifold (M, ω) which is called even (odd) iff the non-degenerate closed 2-form ω is even (odd), i.e., $\epsilon(\omega) = 0$ ($\epsilon(\omega) = 1$). Let Γ be a symmetric (affine) connection on M (not necessarily preserving ω). Given a point $p \in M$ and, in a vicinity of p , local coordinates $\{x^i, \epsilon(x^i) = \epsilon_i, i = 1, \dots, 2N\}$ then a geodesic $x(t)$ through p is defined by

$$\begin{aligned} \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^k}{dt} \frac{dx^j}{dt} &= 0, \quad x(0) = p, \quad \epsilon(\Gamma^i_{jk}) = \epsilon_i + \epsilon_j + \epsilon_k, \\ \Gamma^i_{jk} &= (-1)^{\epsilon_j \epsilon_k} \Gamma^i_{kj}, \end{aligned} \quad (1)$$

with t being a (real) canonical parameter along that geodesic.

The coordinate system is called *geodesic* (at p) iff $\Gamma^i_{jk}(p) = 0$. This means that for any geodesic coordinates $\{x'^j\}$ it holds $x^i = \delta^i_j x'^j + \phi^i(x')$ where $\phi^i(x')$ together with its first two derivatives vanishes. Let $T_p M$ be the tangent space at $p \in M$. Now, let $v^i := \left(\frac{dx^i}{dt} \right)_{t=0} \in T_p M$ be the tangent on the geodesic at p which, together with $x(0) = p$ uniquely determines the geodesic curve. Then, in the vicinity of p , the coordinate system $\{y^i = v^i t\}$ constitutes a particular class of geodesic coordinates being called *normal*.

In normal coordinates the geodesic equation (1) reduces,

$$\Gamma^i_{jk}(y) y^k y^j = 0, \quad (2)$$

where $\Gamma^i_{jk}(y)$ are the Christoffel symbols being calculated at the point p in terms of the normal coordinates y^i . Conversely, for any coordinate system $\{y^i\}$ obeying (2) any geodesic, due to

Eq. (1), satisfies $d^2 y^i / dt^2 = 0$ and, hence, $\{y^i = v^i t\}$ with some v^i .³ Also in case of supermanifolds, like in ordinary manifolds, it holds that under an arbitrary analytic transformation of the coordinates $x^i \rightarrow x'^i$ the corresponding normal coordinates undergo a linear homogeneous transformation with constant coefficients a^i_j ,

$$y'^i = a^i_j y^j. \quad (3)$$

In normal coordinates many of the geometric properties of the (super)manifold are much easier to derive than in arbitrary coordinates.⁴ This is the reason for using them in the following.

Obviously, normal coordinates are defined by the connection (and do not depend on ω). However, the defining identities (2) can be rewritten in the equivalent form with the help of ω ,

$$\Gamma_{ijk}(y) y^k y^j \equiv 0, \quad (4)$$

where, independent of the chosen coordinate system,

$$\begin{aligned} \Gamma_{ijk} &= \omega_{il} \Gamma^l_{jk}, & \Gamma^i_{jk} &= \omega^{il} \Gamma_{ljk} (-1)^{\epsilon_l + \epsilon(\omega)(\epsilon_i + \epsilon_l)}, \\ \epsilon(\Gamma_{ijk}) &= \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k. \end{aligned} \quad (5)$$

and

$$\omega^{ik} \omega_{kj} (-1)^{\epsilon_k + \epsilon(\omega)(\epsilon_i + \epsilon_k)} = \delta^i_j, \quad \omega_{ik} \omega^{kj} (-1)^{\epsilon_i + \epsilon(\omega)(\epsilon_i + \epsilon_k)} = \delta^j_i, \quad (6)$$

It follows from (4) and the symmetry properties of Γ_{ijk} w.r.t. $(j\ k)$ that

$$\Gamma_{ijk}(0) = 0. \quad (7)$$

In normal coordinates there exist additional relations at p containing the partial derivatives of Γ_{ijk} . Namely, consider the Taylor expansion of $\Gamma_{ijk}(y)$ at $y = 0$,

$$\Gamma_{ijk}(y) = \sum_{n=1}^{\infty} \frac{1}{n!} A_{ijkj_1 \dots j_n} y^{j_n} \dots y^{j_1}, \quad (8)$$

where

$$A_{ijkj_1 \dots j_n} = A_{ijkj_1 \dots j_n}(p) = \left. \frac{\partial_r^n \Gamma_{ijk}}{\partial y^{j_1} \dots \partial y^{j_n}} \right|_{y=0} \quad (9)$$

is called an *affine extension* of Γ_{ijk} of order $n = 1, 2, \dots$. The symmetry properties of $A_{ijkj_1 \dots j_n}$ are evident from their definition (9), namely, they are (generalized) symmetric w.r.t. $(j\ k)$ as well as $(j_1 \dots j_n)$.

The set of all affine extensions of Γ_{ijk} uniquely defines a symmetric connection according to (8). In terms of $A_{ijkj_1 \dots j_n}$ the (defining) property (4) can be represented by the following

³Here, we generalized to the case of supermanifolds the introduction of normal coordinates as has been given for affine manifolds for the first time by Veblen, see, e.g., [8]. However, also the more modern definition [9] using the exponential map $\exp_p : U \rightarrow M$ from a small neighborhood U of $0 \in T_p M$ onto M with the property $\exp_p(v) = x(1)$ could have been generalized. Thereby, the exponential map on the supermanifold M has to be defined according to, e.g., DeWitt [6].

⁴Remind that in normal coordinates the equations of geodesics through the origin (at p) have the same form as the equations of straight lines in Euclidean space in cartesian coordinates.

sequence of identities:

$$\begin{aligned}
& A_{ijl i_1 \dots i_n} + \sum_{k=1}^n A_{ij i_k l i_1 \dots i_{k-1} i_{k+1} \dots i_n} (-1)^{\epsilon_{i_k} (\epsilon_l + \epsilon_{i_1} + \dots + \epsilon_{i_{k-1}})} \\
& + \sum_{k=1}^n A_{il i_k j i_1 \dots i_{k-1} i_{k+1} \dots i_n} (-1)^{\epsilon_j (\epsilon_l + \epsilon_{i_k}) + \epsilon_{i_k} (\epsilon_l + \epsilon_{i_1} + \dots + \epsilon_{i_{k-1}})} \\
& + \sum_{k=1}^{n-1} \sum_{m=k+1}^n A_{i i_k i_m j l i_1 \dots i_{k-1} i_{k+1} \dots i_{m-1} i_{m+1} \dots i_n} (-1)^{(\epsilon_{i_k} + \epsilon_{i_m}) (\epsilon_j + \epsilon_l + \epsilon_{i_1} + \dots + \epsilon_{i_{k-1}}) + \epsilon_{i_m} (\epsilon_{i_{k+1}} + \dots + \epsilon_{i_{m+1}})} = 0.
\end{aligned} \tag{10}$$

For given n these identities contain $P = 1 + 2n + n(n-1)/2 = (n+2)(n+1)/2$ terms. For example, when $n = 1$ we have $P = 3$ terms:

$$A_{ijkl} + A_{ijlk} (-1)^{\epsilon_k \epsilon_l} + A_{iklj} (-1)^{\epsilon_j (\epsilon_l + \epsilon_k)} = 0, \tag{11}$$

and for $n = 2$ we have $P = 6$ terms:

$$\begin{aligned}
& A_{ijklm} + A_{ijlkm} (-1)^{\epsilon_k \epsilon_l} + A_{ikljm} (-1)^{\epsilon_j (\epsilon_l + \epsilon_k)} \\
& + A_{ijmkl} (-1)^{\epsilon_m (\epsilon_l + \epsilon_k)} + A_{ilmjk} (-1)^{(\epsilon_j + \epsilon_k) (\epsilon_m + \epsilon_l)} + A_{ikmjl} (-1)^{\epsilon_j (\epsilon_m + \epsilon_k) + \epsilon_m \epsilon_l} = 0.
\end{aligned} \tag{12}$$

Analogously, the affine extensions of an arbitrary tensor $T = (T^{i_1 \dots i_k}_{m_1 \dots m_l})$ on M are defined as tensors on M whose components at $p \in M$ in the local coordinates (x^1, \dots, x^{2N}) are given by the formula

$$T^{i_1 \dots i_k}_{m_1 \dots m_l, j_1 \dots j_n} \equiv T^{i_1 \dots i_k}_{m_1 \dots m_l, j_1 \dots j_n}(0) = \left. \frac{\partial^n T^{i_1 \dots i_k}_{m_1 \dots m_l}}{\partial y^{j_1} \dots \partial y^{j_n}} \right|_{y=0} \tag{13}$$

where (y^1, \dots, y^{2N}) are normal coordinates associated with (x^1, \dots, x^{2N}) at p . The first extension of any tensor coincides with its covariant derivative because $\Gamma^i_{jk}(0) = 0$ in normal coordinates.

In the following, any relation containing affine extensions are to be understood as holding in a neighborhood U of an arbitrary point $p \in M$. Let us also observe the convention that, if a relations holds for arbitrary local coordinates, the arguments of the related quantities will be suppressed.

3 First order affine extension of Christoffel symbols and curvature tensor of Fedosov supermanifolds

Suppose we are given an even (odd) Fedosov supermanifold (M, ω, Γ) , $\epsilon(\omega) = 0$, $(\epsilon(\omega) = 1)$. This means that the symmetric connection Γ (or, equivalently, the covariant derivative ∇) respects the symplectic structure ω : $\omega \nabla = 0$. In local coordinates (x) this condition reads

$$\omega_{ij,k} = \Gamma_{ijk} - \Gamma_{jik} (-1)^{\epsilon_i \epsilon_j}. \tag{14}$$

The curvature tensor $R_{ijkl}(x)$, $\epsilon(R_{ijkl}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l$ of an even (odd) symplectic connection Γ has the following representation (see Ref. [1]),

$$\begin{aligned}
R_{ijkl} &= -\omega_{in} \Gamma^n_{jk,l} + \omega_{in} \Gamma^n_{jl,k} (-1)^{\epsilon_k \epsilon_l} + \Gamma_{ikn} \Gamma^n_{jl} (-1)^{\epsilon_k \epsilon_j} - \Gamma_{iln} \Gamma^n_{jk} (-1)^{\epsilon_l (\epsilon_j + \epsilon_k)} \\
&= -\Gamma_{ijk,l} + \Gamma_{ijl,k} (-1)^{\epsilon_k \epsilon_l} + \Gamma_{nik} \Gamma^n_{jl} (-1)^{\epsilon_n \epsilon_i + \epsilon_k (\epsilon_j + \epsilon_n)} - \Gamma_{nil} \Gamma^n_{jk} (-1)^{\epsilon_n \epsilon_i + \epsilon_l (\epsilon_k + \epsilon_j + \epsilon_n)},
\end{aligned} \tag{15}$$

and obeys the following symmetry properties

$$R_{ijkl} = -(-1)^{\epsilon_k \epsilon_l} R_{ijlk}, \quad R_{ijkl} = (-1)^{\epsilon_i \epsilon_j} R_{jikl}; \quad (16)$$

in deriving (15) the following relations, due to (5) and (6),

$$\begin{aligned} \Gamma_{jk,l}^n &= (-1)^{\epsilon_m + \epsilon(\omega)(\epsilon_m + \epsilon_n)} (\omega^{nm} \Gamma_{mjk,l} + \omega^{nm},_l \Gamma_{mjk} (-1)^{\epsilon_l(\epsilon(\omega) + \epsilon_k + \epsilon_j + \epsilon_m)}) , \quad (17) \\ \omega_{in} \omega^{nm},_l (-1)^{\epsilon(\omega)\epsilon_n} &= \omega_{in,l} \omega^{nm} (-1)^{\epsilon_l(\epsilon(\omega) + \epsilon_n + \epsilon_m) + \epsilon(\omega)\epsilon_n} . \quad (18) \end{aligned}$$

were used together with the Eq. (14).

As has been shown in Ref. [1], for the symplectic curvature tensor R_{ijkl} there holds the (super) Jacobi identity,

$$R_{ijkl}(-1)^{\epsilon_j \epsilon_l} + R_{iljk}(-1)^{\epsilon_l \epsilon_k} + R_{iklj}(-1)^{\epsilon_k \epsilon_j} = 0, \quad (19)$$

and the first Bianchi identity containing a cyclic permutation of all the indices,

$$R_{ijkl}(-1)^{\epsilon_i \epsilon_l} + R_{lijk}(-1)^{\epsilon_l \epsilon_k + \epsilon_l \epsilon_j} + R_{klji}(-1)^{\epsilon_k \epsilon_j + \epsilon_l \epsilon_j + \epsilon_i \epsilon_k} + R_{jkli}(-1)^{\epsilon_i \epsilon_j + \epsilon_i \epsilon_k} = 0. \quad (20)$$

It follows from (14) that among the affine extensions of ω_{ij} and Γ_{ijk} there must exist some relations. To this end let us consider the Taylor expansion of ω_{ij} in the normal coordinates (y^1, \dots, y^{2N}) at $p \in M$,

$$\omega_{ij}(y) = \sum_{n=1}^{\infty} \frac{1}{n!} \Omega_{ij,j_1 \dots j_n} y^{j_n} \dots y^{j_1}, \quad \Omega_{ij,j_1 \dots j_n} = \omega_{ij,j_1 \dots j_n}(0). \quad (21)$$

Using the symmetry properties of $\omega_{ij,j_1 \dots j_n}(0)$ one can easily obtain the following Taylor expansion for $\omega_{ij,k}$

$$\omega_{ij,k}(y) = \sum_{n=1}^{\infty} \frac{1}{n!} \Omega_{ij,kj_1 \dots j_n} y^{j_n} \dots y^{j_1}. \quad (22)$$

Taking into account (14) and comparing (8) and (22) we obtain

$$\Omega_{ij,kj_1 \dots j_n} = A_{ijkj_1 \dots j_n} - A_{jikj_1 \dots j_n} (-1)^{\epsilon_i \epsilon_j}. \quad (23)$$

In particular,

$$\Omega_{ij,kl} = A_{ijkl} - A_{jikl} (-1)^{\epsilon_i \epsilon_j}. \quad (24)$$

Now, consider the curvature tensor R_{ijkl} , Eq. (15), in the normal coordinates at $p \in M$. Then, due to $\Gamma_{ijk}(p) = 0$, we obtain the following representation of the curvature tensor in terms of the affine extensions of the symplectic connection

$$R_{ijkl}(0) = -A_{ijkl} + A_{ijlk} (-1)^{\epsilon_k \epsilon_l}. \quad (25)$$

Taking into account (11) and (25) the existence of some relation containing the curvature tensor and the second affine extension of ω can be expected. Indeed, step by step one obtains

$$\begin{aligned} R_{iklj}(0) &= A_{ikjl} (-1)^{\epsilon_l \epsilon_j} - A_{iklj} = A_{ijkl} (-1)^{(\epsilon_l + \epsilon_k) \epsilon_j} - A_{iklj} \\ &= A_{ijkl} (-1)^{(\epsilon_l + \epsilon_k) \epsilon_j} + A_{ikjl} (-1)^{\epsilon_l \epsilon_j} + A_{iljk} (-1)^{(\epsilon_l + \epsilon_j) \epsilon_k} \\ &= 2A_{ijkl} (-1)^{(\epsilon_l + \epsilon_k) \epsilon_j} + A_{iljk} (-1)^{(\epsilon_l + \epsilon_j) \epsilon_k} \\ &= 2A_{ijkl} (-1)^{(\epsilon_l + \epsilon_k) \epsilon_j} + A_{ijlk} (-1)^{(\epsilon_l + \epsilon_j) \epsilon_k + \epsilon_j \epsilon_l} \\ &= 2A_{ijkl} (-1)^{(\epsilon_l + \epsilon_k) \epsilon_j} + [R_{ijkl}(0) + A_{ijkl}] (-1)^{(\epsilon_l + \epsilon_k) \epsilon_j} \\ &= 3A_{ijkl} (-1)^{(\epsilon_l + \epsilon_k) \epsilon_j} + R_{ijkl}(0) (-1)^{(\epsilon_l + \epsilon_k) \epsilon_j}. \quad (26) \end{aligned}$$

Therefore we have the representation of the first order affine extension of the Christoffel symbols in terms of the curvature tensor at p ,

$$A_{ijkl} \equiv \Gamma_{ijk,l}(0) = -\frac{1}{3} [R_{ijkl}(0) + R_{ikjl}(0)(-1)^{\epsilon_k \epsilon_j}], \quad (27)$$

where the symmetry properties (16) of the curvature tensor were used.

Notice, that relation (27) was derived in normal coordinates. It seems to be of general interest to find its analog relation in terms arbitrary local coordinates (x) because the Christoffel symbols are not tensors while the r.h.s. of (27) is a tensor. To this end let us relate the arbitrary local coordinates to the normal ones as follows [8],

$$x^i = x_0^i + y^i - \frac{1}{2} (\Gamma^i_{jk})_0 y^k y^j + \dots + \frac{1}{n!} \left(\frac{\partial_r^n x^i}{\partial y^{j_1} \dots \partial y^{j_n}} \right)_0 y^{j_n} \dots y^{j_1} + \dots, \quad (28)$$

where the subscript 0 indicates that the corresponding quantity is taken at the point $p \in M$. As a consequence we have

$$\left(\frac{\partial_r x^i}{\partial y^j} \right)_0 = \delta_j^i. \quad (29)$$

Since the Jacobian at p is different of zero the series (28) can be inverted, $y^i = x^i - x_0^i + \phi^i(x - x_0)$, where $\phi^i(x - x_0)$ depends on the second and higher powers of the differences $x^j - x_0^j$.

Under that change of coordinates ($x \rightarrow y$) in some vicinity U of p the Christoffel symbols Γ^i_{jk} transform according to the rule

$$\Gamma^i_{jk}(y) = \frac{\partial_r y^i}{\partial x^l} \left(\Gamma^l_{mn}(x) \frac{\partial_r x^n}{\partial y^k} \frac{\partial_r x^m}{\partial y^j} (-1)^{\epsilon_k(\epsilon_j + \epsilon_m)} + \frac{\partial_r^2 x^l}{\partial y^j \partial y^k} \right). \quad (30)$$

These relations can be rewritten in the form

$$\Gamma_{ijk}(y) = \left(\Gamma_{pqr}(x) \frac{\partial_r x^r}{\partial y^k} \frac{\partial_r x^q}{\partial y^j} (-1)^{\epsilon_k(\epsilon_j + \epsilon_q)} + \omega_{pq}(x) \frac{\partial_r^2 x^q}{\partial y^j \partial y^k} \right) \frac{\partial_r x^p}{\partial y^i} (-1)^{(\epsilon_k + \epsilon_j)(\epsilon_i + \epsilon_p)}, \quad (31)$$

where we used the definition (5) of Γ_{ijk} and the transformation rule of the symplectic structure,

$$\omega_{ij}(y) = \omega_{pq}(x) \frac{\partial_r x^q}{\partial y^j} \frac{\partial_r x^p}{\partial y^i} (-1)^{\epsilon_j(\epsilon_p + \epsilon_i)} \quad (32)$$

From (30) one can express the matrix of second derivatives in the form

$$\frac{\partial_r^2 x^q}{\partial y^j \partial y^k} = \frac{\partial_r x^q}{\partial y^l} \Gamma^l_{jk}(y) - \Gamma^q_{lm}(x) \frac{\partial_r x^m}{\partial y^k} \frac{\partial_r x^l}{\partial y^j} (-1)^{\epsilon_k(\epsilon_j + \epsilon_l)}. \quad (33)$$

In particular at $p \in M$ ($y = 0$), observing the equality (29) we have the relation

$$\left(\frac{\partial_r^2 x^q}{\partial y^j \partial y^k} \right)_0 = -\Gamma^q_{lm}(x_0) \left(\frac{\partial_r x^m}{\partial y^k} \right)_0 \left(\frac{\partial_r x^l}{\partial y^j} \right)_0 (-1)^{\epsilon_k(\epsilon_j + \epsilon_l)} \equiv -\Gamma^p_{jk}(x_0). \quad (34)$$

Differentiating (31) with respect to y we find

$$\begin{aligned} \Gamma_{ijk,l}(y) &= \Gamma_{pqr;s}(x) \frac{\partial_r x^s}{\partial y^l} \frac{\partial_r x^r}{\partial y^k} \frac{\partial_r x^q}{\partial y^j} \frac{\partial_r x^p}{\partial y^i} (-1)^{(\epsilon_j + \epsilon_k + \epsilon_l)(\epsilon_i + \epsilon_p) + (\epsilon_k + \epsilon_l)(\epsilon_j + \epsilon_q) + \epsilon_l(\epsilon_k + \epsilon_r)} \\ &+ \omega_{pq,r}(x) \frac{\partial_r x^r}{\partial y^l} \frac{\partial_r^2 x^q}{\partial y^j \partial y^k} \frac{\partial_r x^p}{\partial y^i} (-1)^{(\epsilon_k + \epsilon_j)(\epsilon_i + \epsilon_p) + \epsilon_l(\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_q + \epsilon_p)} \\ &+ \omega_{pq}(x) \frac{\partial_r^2 x^q}{\partial y^j \partial y^k} \frac{\partial_r^2 x^p}{\partial y^i \partial y^l} (-1)^{(\epsilon_k + \epsilon_j)(\epsilon_i + \epsilon_p)} + \omega_{pq}(x) \frac{\partial_r^3 x^q}{\partial y^j \partial y^k \partial y^l} \frac{\partial_r x^p}{\partial y^i} (-1)^{(\epsilon_j + \epsilon_k + \epsilon_l)(\epsilon_i + \epsilon_p)}, \end{aligned}$$

where the covariant derivative (for arbitrary local coordinates) is defined by

$$\Gamma_{pqr;s} = \Gamma_{pqr,s} - \Gamma_{pqn}\Gamma_{rs}^n - \Gamma_{pnr}\Gamma_{qs}^n(-1)^{\epsilon_r(\epsilon_n+\epsilon_q)} - \Gamma_{nqr}\Gamma_{ps}^n(-1)^{(\epsilon_r+\epsilon_q)(\epsilon_n+\epsilon_p)}. \quad (35)$$

Now, making use of Eqs. (14) and (5), and restricting to the point $p \in M$, i.e., taking $y = 0$, we get

$$\Gamma_{ijk,l}(0) = (\Gamma_{ijk;l}(x_0) - \Gamma_{iln}(x_0)\Gamma_{jk}^n(x_0)(-1)^{\epsilon_l(\epsilon_j+\epsilon_k)}) + \omega_{iq}(x_0) \left(\frac{\partial_r^3 x^q}{\partial y^j \partial y^k \partial y^l} \right)_0. \quad (36)$$

Due to (36) and the identity (11),

$$\Gamma_{ijk,l}(0) + \Gamma_{ijl,k}(0)(-1)^{\epsilon_k\epsilon_l} + \Gamma_{ikl,j}(0)(-1)^{\epsilon_j(\epsilon_l+\epsilon_k)} = 0,$$

the matrix of third derivatives at p obeys the following relation,

$$\omega_{iq}(x_0) \left(\frac{\partial_r^3 x^q}{\partial y^j \partial y^k \partial y^l} \right)_0 = \frac{1}{3} \left[-\Gamma_{ijk;l} - \Gamma_{ijl,k}(-1)^{\epsilon_k\epsilon_l} - \Gamma_{ikl;j}(-1)^{\epsilon_j(\epsilon_k+\epsilon_l)} + \Gamma_{iln}\Gamma_{jk}^n(-1)^{(\epsilon_j+\epsilon_k)\epsilon_l} + \Gamma_{ijn}\Gamma_{kl}^n + \Gamma_{ikn}\Gamma_{jl}^n(-1)^{\epsilon_k\epsilon_j} \right](x_0). \quad (37)$$

With the help of (37) we get the following transformation law for $\Gamma_{ijk;l}$ under change of coordinates at the point p

$$\Gamma_{ijk,l}(0) = \left[\Gamma_{ijk,l}(x_0) - \frac{1}{3} Z_{ijkl}(x_0) \right], \quad (38)$$

with the abbreviation

$$\begin{aligned} Z_{ijkl} = & \Gamma_{ijk;l} + \Gamma_{ijl,k}(-1)^{\epsilon_k\epsilon_l} + \Gamma_{ikl;j}(-1)^{(\epsilon_k+\epsilon_l)\epsilon_j} \\ & + 2\Gamma_{iln}\Gamma_{jk}^n(-1)^{(\epsilon_k+\epsilon_j)\epsilon_l} - \Gamma_{ikn}\Gamma_{jl}^n(-1)^{\epsilon_j\epsilon_k} - \Gamma_{ijn}\Gamma_{kl}^n \end{aligned} \quad (39)$$

In straightforward manner one can check that the relations (36) reproduce the right transformation law for the curvature tensor,

$$\begin{aligned} R_{ijkl}(0) &= -\Gamma_{ijk,l}(0) + \Gamma_{ijl,k}(0)(-1)^{\epsilon_k\epsilon_l} \\ &= \left[-\Gamma_{ijk;l} + \Gamma_{iln}\Gamma_{jk}^n(-1)^{\epsilon_l(\epsilon_j+\epsilon_k)} + \Gamma_{ijl,k}(-1)^{\epsilon_k\epsilon_l} - \Gamma_{ikn}\Gamma_{jl}^n(-1)^{\epsilon_j\epsilon_k} \right](x_0) \\ &= \left(R_{pqrs}(x) \frac{\partial_r x^s}{\partial y^l} \frac{\partial_r x^r}{\partial y^k} \frac{\partial_r x^q}{\partial y^j} \frac{\partial_r x^p}{\partial y^i} (-1)^{(\epsilon_j+\epsilon_k+\epsilon_l)(\epsilon_i+\epsilon_p)+(\epsilon_k+\epsilon_l)(\epsilon_j+\epsilon_q)+\epsilon_l(\epsilon_k+\epsilon_r)} \right)_{x=x_0} \\ &= R_{ijkl}(x_0), \end{aligned} \quad (40)$$

where we have taken into account the following relation

$$\begin{aligned} & -\Gamma_{pqs;r} + \Gamma_{prn}\Gamma_{qs}^n(-1)^{\epsilon_r(\epsilon_q+\epsilon_s)} + \Gamma_{pqr;s}(-1)^{\epsilon_r\epsilon_s} - \Gamma_{psn}\Gamma_{qr}^n(-1)^{\epsilon_s\epsilon_q} \\ &= -\Gamma_{pqs,r} + \Gamma_{pqn}\Gamma_{sr}^n + \Gamma_{pns}\Gamma_{qr}^n(-1)^{\epsilon_s(\epsilon_n+\epsilon_q)} + \Gamma_{nqs}\Gamma_{pr}^n(-1)^{(\epsilon_s+\epsilon_q)(\epsilon_n+\epsilon_p)} \\ & \quad + \Gamma_{prn}\Gamma_{qs}^n(-1)^{\epsilon_r(\epsilon_q+\epsilon_s)} + \Gamma_{pqr,s}(-1)^{\epsilon_r\epsilon_s} - \Gamma_{pqn}(\Gamma_{rs}^n(-1)^{\epsilon_r\epsilon_s} \\ & \quad - \Gamma_{pnr}\Gamma_{qs}^n(-1)^{\epsilon_r(\epsilon_n+\epsilon_q)+\epsilon_r\epsilon_s} - \Gamma_{nqr}\Gamma_{ps}^n(-1)^{(\epsilon_r+\epsilon_q)(\epsilon_n+\epsilon_p)+\epsilon_r\epsilon_s} - \Gamma_{psn}\Gamma_{qr}^n(-1)^{\epsilon_s\epsilon_q} \\ &= -\Gamma_{pqs,r} + \Gamma_{nqs}\Gamma_{pr}^n(-1)^{(\epsilon_s+\epsilon_q)(\epsilon_n+\epsilon_p)} + \Gamma_{pqr,s}(-1)^{\epsilon_r\epsilon_s} - \Gamma_{nqr}\Gamma_{ps}^n(-1)^{(\epsilon_r+\epsilon_q)(\epsilon_n+\epsilon_p)+\epsilon_r\epsilon_s} \\ &= -\Gamma_{pqs,r} + \Gamma_{pqr,s}(-1)^{\epsilon_r\epsilon_s} - \Gamma_{npr}\Gamma_{qs}^n(-1)^{\epsilon_r(\epsilon_n+\epsilon_q+\epsilon_s)+\epsilon_n\epsilon_p} + \Gamma_{nps}\Gamma_{qr}^n(-1)^{\epsilon_s(\epsilon_n+\epsilon_q)+\epsilon_n\epsilon_p} \\ &= R_{pqsr}. \end{aligned}$$

The relations above are derived at p . However, since $p \in M$ is arbitrary the following relations hold in any local coordinate system (x) : the curvature tensor reads in terms of covariant derivatives of Christoffel symbols,

$$R_{ijkl} = -\Gamma_{ijk;l} + \Gamma_{iln}\Gamma^n_{jk}(-1)^{\epsilon_l(\epsilon_j+\epsilon_k)} + \Gamma_{ijl;k}(-1)^{\epsilon_k\epsilon_l} - \Gamma_{ikn}\Gamma^n_{jl}(-1)^{\epsilon_j\epsilon_k}, \quad (41)$$

and from (38) and (40) it follows that the relation (27) is to be generalized as

$$\Gamma_{ijk;l} - \frac{1}{3}Z_{ijkl} = -\frac{1}{3}[R_{ijkl} + R_{ikjl}(-1)^{\epsilon_j\epsilon_k}]. \quad (42)$$

The last equation gets an identity when using the definition (39) of $Z_{ijkl}(x)$ and relation (41) for $R_{ijkl}(x)$ on the r.h.s..

4 Second order affine extension of symplectic structure and curvature tensor on Fedosov supermanifolds

Now, let us consider the relation between the second order affine extension of symplectic structure and the symplectic curvature tensor. It is easily found by taking into account (24) and (27). Indeed we obtain

$$\begin{aligned} \Omega_{ij,kl} &= A_{ijkl} - A_{jikl}(-1)^{\epsilon_i\epsilon_j} \\ &= -\frac{1}{3}[(R_{ijkl}(0) + R_{ikjl}(0)(-1)^{\epsilon_k\epsilon_j}) - (R_{jikl}(0)(-1)^{\epsilon_i\epsilon_j} + R_{jkil}(0)(-1)^{\epsilon_i(\epsilon_j+\epsilon_k)})] \\ &= -\frac{1}{3}[R_{ikjl}(0)(-1)^{\epsilon_k\epsilon_j} - R_{jkil}(0)(-1)^{\epsilon_i(\epsilon_j+\epsilon_k)}] \\ &= -\frac{1}{3}[R_{kilj}(0)(-1)^{\epsilon_i\epsilon_j} + R_{kjil}(0)(-1)^{\epsilon_l\epsilon_j}](-1)^{(\epsilon_i+\epsilon_j)\epsilon_k+(\epsilon_i+\epsilon_l)\epsilon_j} \\ &= \frac{1}{3}R_{klji}(0)(-1)^{\epsilon_l\epsilon_i}(-1)^{(\epsilon_i+\epsilon_j)\epsilon_k+(\epsilon_i+\epsilon_l)\epsilon_j}, \end{aligned} \quad (43)$$

where in the last line the Jacobi identity (19) has been used for the curvature tensor. Then from (43) it follows

$$\omega_{ij,kl}(0) = \frac{1}{3}R_{klji}(0)(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}. \quad (44)$$

Obviously, Eq. (44) holds at $p \in M$ in normal coordinates (y) . However, it can easily be generalized to arbitrary local coordinates (x) . One only has to remind that Eq. (42) is the generalization of Eq. (27) for $\Gamma_{ijk,l}(0)$ in order to obtain

$$\begin{aligned} \omega_{ij,kl}(0) &= \Gamma_{ijk,l}(0) - \Gamma_{jik,l}(0)(-1)^{\epsilon_i\epsilon_j} \\ &= \omega_{ij,k;l}(x_0) - \frac{1}{3}Z_{ijkl}(x_0) + \frac{1}{3}Z_{ijlk}(x_0)(-1)^{\epsilon_i\epsilon_j}, \end{aligned} \quad (45)$$

with

$$\omega_{ij,k;l} = \Gamma_{ijk;l} - \Gamma_{jik;l}(-1)^{\epsilon_i\epsilon_j}, \quad (46)$$

as a consequence of Eq. (14). Thereby, the first derivative of ω_{ij} is understood as usual partial one while the second one is the covariant derivative $\omega_{ij,k;l} \equiv (\omega_{ij,k})\nabla_l$. Again, since $p \in M$ is arbitrary, we finally obtain the desired generalization of Eq. (44):

$$\omega_{ij,k;l} - \frac{1}{3}[Z_{ijkl} - Z_{jikl}(-1)^{\epsilon_i\epsilon_j}] = \frac{1}{3}(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}R_{klji}. \quad (47)$$

5 Third order affine extension of symplectic structure and first order affine extension of curvature tensor

First, note that, because of $\omega_{ij;k} = 0$ and $\omega_{ln}R^n_{mjk;i} = (\omega_{ln}R^n_{mjk})_{;i} = R_{lmjk;i}$, on any Fedosov supermanifolds the second Bianchi identity exists [1],

$$R_{ijkl;m}(-1)^{\epsilon_m\epsilon_k} + R_{ijlm;k}(-1)^{\epsilon_l\epsilon_k} + R_{ijmk;l}(-1)^{\epsilon_m\epsilon_l} \equiv 0. \quad (48)$$

However, there exist further relations among the affine extensions of Γ_{ijk} , R_{ijkl} and ω_{ij} on a Fedosov supermanifold. In order to show them, let us rewrite R_{ijkl} in terms of Γ_{ijk} and their first derivatives:

$$\begin{aligned} R_{ijkl} = & -\Gamma_{ijk,l} + \Gamma_{ijl,k}(-1)^{\epsilon_k\epsilon_l} \\ & + (\Gamma_{ikr}\omega^{rs}\Gamma_{sjl}(-1)^{\epsilon_j\epsilon_k} - \Gamma_{ilr}\omega^{rs}\Gamma_{sjk}(-1)^{\epsilon_l(\epsilon_j+\epsilon_k)}) (-1)^{\epsilon_r+\epsilon(\omega)(\epsilon_r+\epsilon_s)} \\ & - (\omega_{ir}\omega^{rs}{}_{,l}\Gamma_{sjk}(-1)^{\epsilon_l(\epsilon_j+\epsilon_k+\epsilon_s)} - \omega_{ir}\omega^{rs}{}_{,j}\Gamma_{sjl}(-1)^{\epsilon_k(\epsilon_j+\epsilon_s)}) (-1)^{\epsilon_s+\epsilon(\omega)(\epsilon_r+\epsilon_s)}. \end{aligned} \quad (49)$$

Differentiating both sides w.r.t. y^m , taking the limit $y \rightarrow 0$ and observing that, because of $\Gamma_{ijk}(0) = 0$, the first extension of the symplectic structure vanishes, $\omega_{ij,k}(0) = 0$, we have

$$R_{ijkl,m}(0) = -A_{ijklm} + A_{ijlkm}(-1)^{\epsilon_l\epsilon_k}. \quad (50)$$

That relation will be used to eliminate within the relation (12) all the extensions of the Christoffel symbols in favor of A_{ijklm} and corresponding derivations of symplectic curvature tensor according to the following relations:

$$\begin{aligned} A_{ijlkm}(-1)^{\epsilon_l\epsilon_k} &= R_{ijkl,m}(0) + A_{ijklm}, \\ A_{ijmkl}(-1)^{\epsilon_m\epsilon_k} &= R_{ijkm,l}(0) + A_{ijkm,l} = R_{ijkm,l}(0) + A_{ijlkm}(-1)^{\epsilon_l\epsilon_m}, \\ A_{ikljm}(-1)^{\epsilon_j\epsilon_l} &= R_{ikjl,m}(0) + A_{ikjl,m} = R_{ikjl,m}(0) + A_{ijlkm}(-1)^{\epsilon_j\epsilon_k}, \\ A_{ikmj,l}(-1)^{\epsilon_j\epsilon_m} &= R_{ikjm,l}(0) + A_{ikjm,l} = R_{ikjm,l}(0) + A_{ijlkm}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l}, \\ A_{ilmjk}(-1)^{\epsilon_j\epsilon_m} &= R_{iljm,k}(0) + A_{iljm,k} = R_{iljm,k}(0) + A_{ijlkm}(-1)^{\epsilon_j\epsilon_l+\epsilon_m\epsilon_k} \\ &= R_{iljm,k}(0) + R_{ijkl,m}(0)(-1)^{\epsilon_j\epsilon_l+\epsilon_m\epsilon_k+\epsilon_l\epsilon_k} + A_{ijklm}(-1)^{\epsilon_j\epsilon_l+\epsilon_m\epsilon_k+\epsilon_l\epsilon_k}. \end{aligned}$$

Putting them into the identity (12) we obtain

$$\begin{aligned} 6A_{ijklm} + 2R_{ijkl,m}(0) + R_{ijkm,l}(0)(-1)^{\epsilon_m\epsilon_l} + R_{ikjl,m}(0)(-1)^{\epsilon_j\epsilon_k} \\ + R_{ikjm,l}(0)(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,k}(0)(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_m+\epsilon_l)} = 0, \end{aligned} \quad (51)$$

and we obtain the following representation of the second order affine extension of Γ_{ijk} in terms of first order derivatives of the curvature tensor

$$\begin{aligned} A_{ijklm} = & -\frac{1}{6} \left[2R_{ijkl,m}(0) + R_{ijkm,l}(0)(-1)^{\epsilon_m\epsilon_l} + R_{ikjl,m}(0)(-1)^{\epsilon_j\epsilon_k} \right. \\ & \left. + R_{ikjm,l}(0)(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,k}(0)(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_m+\epsilon_l)} \right]. \end{aligned} \quad (52)$$

In its turn from (23) we have

$$\omega_{ik,jlm}(0) = A_{ikjlm} - A_{kijlm}(-1)^{\epsilon_i\epsilon_k}, \quad (53)$$

and therefore we get

$$\begin{aligned} \omega_{ij,klm}(0) = & -\frac{1}{6} \left[R_{ikjl,m}(0)(-1)^{\epsilon_j\epsilon_k} + R_{ikjm,l}(0)(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,k}(0)(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_m)} \right. \\ & \left. - R_{jkim,l}(0)(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)} - R_{jkim,l}(0)(-1)^{\epsilon_m\epsilon_l+\epsilon_i(\epsilon_j+\epsilon_k)} - R_{jlim,k}(0)(-1)^{\epsilon_k(\epsilon_m+\epsilon_l)+\epsilon_i(\epsilon_j+\epsilon_l)} \right] \end{aligned} \quad (54)$$

as the representation of the third order affine extensions of ω_{ij} in terms of the first order affine extension of the symplectic curvature tensor.

Now, we consider the consequences which follow from the symmetry properties of $\omega_{ik,jlm}$,

$$\omega_{ik,jlm} = \omega_{ik,ljm}(-1)^{\epsilon_l \epsilon_j} \quad (55)$$

and the possibility to express A_{ijklm} in terms of the first order derivative of the curvature tensor (52). We have

$$\begin{aligned} 0 &= \omega_{ik,jlm} - \omega_{ik,ljm}(-1)^{\epsilon_l \epsilon_j} \\ &= A_{ikjlm} - A_{kijlm}(-1)^{\epsilon_i \epsilon_k} - (-1)^{\epsilon_l \epsilon_j} (A_{ikljm} - A_{kiljm}(-1)^{\epsilon_i \epsilon_k}) \\ &= A_{ijklm}(-1)^{\epsilon_j \epsilon_k} - A_{kjilm}(-1)^{\epsilon_i(\epsilon_k + \epsilon_j)} - A_{ilkjm}(-1)^{\epsilon_l(\epsilon_k + \epsilon_j)} + A_{klijm}(-1)^{\epsilon_i \epsilon_k + \epsilon_l(\epsilon_i + \epsilon_j)}. \end{aligned} \quad (56)$$

Using Eqs. (53) and the symmetry properties of the curvature tensor we can be rewritten as

$$\begin{aligned} 0 &= 2R_{ijkl,m}(0)(-1)^{\epsilon_j \epsilon_k} - 2R_{kjil,m}(0)(-1)^{\epsilon_i(\epsilon_j + \epsilon_k)} - 2R_{ilkj,m}(0)(-1)^{\epsilon_l(\epsilon_j + \epsilon_k)} \\ &\quad + 2R_{klji,m}(0)(-1)^{\epsilon_i(\epsilon_l + \epsilon_k) + \epsilon_l \epsilon_j} + R_{ijkm,l}(0)(-1)^{\epsilon_k \epsilon_j + \epsilon_m \epsilon_l} - R_{kjim,l}(0)(-1)^{\epsilon_i(\epsilon_k + \epsilon_j) + \epsilon_m \epsilon_l} \\ &\quad - R_{ilkm,j}(0)(-1)^{\epsilon_j(\epsilon_m + \epsilon_l) + \epsilon_k \epsilon_l} + R_{klim,j}(0)(-1)^{\epsilon_j(\epsilon_m + \epsilon_l) + \epsilon_i(\epsilon_k + \epsilon_l)} \\ &\quad + R_{iljm,k}(0)(-1)^{\epsilon_j(\epsilon_k + \epsilon_l) + \epsilon_k(\epsilon_m + \epsilon_l)} - R_{kljm,i}(0)(-1)^{\epsilon_j(\epsilon_i + \epsilon_l) + \epsilon_i(\epsilon_k + \epsilon_l + \epsilon_m)} \\ &\quad - R_{ijlm,k}(0)(-1)^{\epsilon_k(\epsilon_j + \epsilon_m + \epsilon_k)} + R_{kjlm,i}(0)(-1)^{\epsilon_i(\epsilon_j + \epsilon_m + \epsilon_k + \epsilon_l)}. \end{aligned} \quad (57)$$

Due to the first Bianchi identities (20) for the curvature tensor the first four terms in (57) vanish:

$$\begin{aligned} &R_{ijkl,m}(-1)^{\epsilon_j \epsilon_k} - R_{kjil,m}(-1)^{\epsilon_i(\epsilon_j + \epsilon_k)} - R_{ilkj,m}(-1)^{\epsilon_l(\epsilon_j + \epsilon_k)} + R_{klji,m}(-1)^{\epsilon_i(\epsilon_l + \epsilon_k) + \epsilon_l \epsilon_j} \\ &= (-1)^{\epsilon_i \epsilon_l + \epsilon_j \epsilon_k} [R_{ijkl}(-1)^{\epsilon_i \epsilon_l} + R_{jkli}(-1)^{\epsilon_i(\epsilon_j + \epsilon_k)} \\ &\quad + R_{lijk}(-1)^{\epsilon_l(\epsilon_j + \epsilon_k)} + R_{klij}(-1)^{\epsilon_j(\epsilon_k + \epsilon_l) + \epsilon_i \epsilon_k}]_m = 0 \end{aligned}$$

The remaining terms in (57) can be presented in the form

$$\begin{aligned} &[R_{jikm,l}(0)(-1)^{\epsilon_m \epsilon_i} + R_{jkim,l}(0)(-1)^{\epsilon_i \epsilon_k}](-1)^{\epsilon_j(\epsilon_i + \epsilon_k) + \epsilon_m(\epsilon_i + \epsilon_l)} \\ &+ [R_{limk,j}(0)(-1)^{\epsilon_i \epsilon_k} + R_{lkim,j}(0)(-1)^{\epsilon_k \epsilon_m}](-1)^{\epsilon_i(\epsilon_k + \epsilon_l) + \epsilon_j(\epsilon_m + \epsilon_l) + \epsilon_k(\epsilon_i + \epsilon_l)} \\ &+ [R_{iljm,k}(0)(-1)^{\epsilon_m \epsilon_l} + R_{ijml,k}(0)(-1)^{\epsilon_j \epsilon_l}](-1)^{\epsilon_l(\epsilon_k + \epsilon_j) + \epsilon_k(\epsilon_m + \epsilon_j) + \epsilon_m \epsilon_l} \\ &+ [R_{klmj,i}(0)(-1)^{\epsilon_l \epsilon_j} + R_{kjlm,i}(0)(-1)^{\epsilon_j \epsilon_m}](-1)^{\epsilon_i(\epsilon_l + \epsilon_m) + \epsilon_i(\epsilon_k + \epsilon_j) + \epsilon_m \epsilon_j} = 0. \end{aligned} \quad (58)$$

Taking into account the Jacobi identity (19) this equation can be rewritten as

$$\begin{aligned} 0 &= R_{jmik,l}(0)(-1)^{\epsilon_m \epsilon_k + \epsilon_j(\epsilon_i + \epsilon_k) + \epsilon_m(\epsilon_i + \epsilon_l)} + R_{lmki,j}(0)(-1)^{\epsilon_i \epsilon_m + \epsilon_l(\epsilon_i + \epsilon_k) + \epsilon_j(\epsilon_m + \epsilon_l)} \\ &+ R_{imlj,k}(0)(-1)^{\epsilon_m \epsilon_j + \epsilon_l(\epsilon_k + \epsilon_j + \epsilon_m) + \epsilon_k(\epsilon_m + \epsilon_j)} + R_{kmjl,i}(0)(-1)^{\epsilon_l \epsilon_m + \epsilon_i(\epsilon_l + \epsilon_m) + \epsilon_i(\epsilon_k + \epsilon_j) + \epsilon_m \epsilon_j}. \end{aligned} \quad (59)$$

Finally we obtain the following identities for the first affine extension of the curvature tensor

$$\begin{aligned} &R_{mjik,l}(0)(-1)^{\epsilon_j(\epsilon_i + \epsilon_k)} - R_{mijl,k}(0)(-1)^{\epsilon_k(\epsilon_l + \epsilon_j)} \\ &+ R_{mkjl,i}(0)(-1)^{\epsilon_i(\epsilon_j + \epsilon_k + \epsilon_l)} - R_{mlik,j}(0)(-1)^{\epsilon_l(\epsilon_i + \epsilon_j + \epsilon_k)} = 0. \end{aligned} \quad (60)$$

In local coordinates (x) the partial derivatives have to be replaced by the covariant ones and these identities evidently have the form

$$R_{mjik;l}(-1)^{\epsilon_j(\epsilon_i + \epsilon_k)} - R_{mijl;k}(-1)^{\epsilon_k(\epsilon_l + \epsilon_j)} + R_{mkjl;i}(-1)^{\epsilon_i(\epsilon_j + \epsilon_k + \epsilon_l)} - R_{mlik;j}(-1)^{\epsilon_l(\epsilon_i + \epsilon_j + \epsilon_k)} = 0. \quad (61)$$

6 Discussion

In this paper we continued investigation of properties of even and odd Fedosov supermanifolds which we began in [1].

Using normal coordinates on a supermanifold equipped with a symmetric connection we have derived infinite system of equations for affine extension of the Christoffel symbols (10). With the help of these equations we have found relations among the first order affine extensions of the Christoffel symbols and the curvature tensor (27), the second order affine extension of symplectic structure and the curvature tensor (44), the third order affine extension of symplectic structure and the first order affine extension of the curvature tensor (54) existing in normal coordinates on any Fedosov supermanifold. In similar way it is possible to find relations containing higher order affine extensions of symplectic structure, the Christoffel symbols and the curvature tensor. This procedure is closely connected with relations (10), (23), (49). We have established the form of the obtained relations in any local coordinates (see (42), (47)). It was shown the tensor field $\Gamma_{ijk;l}(x) - 1/3Z_{ijkl}(x)$ in terms of what the relations obtained can be presented (38). We have derived a specific identity (61) for the first covariant derivative of the curvature tensor (15).

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